World’s Simplest Poker

Rules. “World’s Simplest Poker” is a two-player poker game in which each player receives one card and there is one *simultaneous* betting round. In particular, the game proceeds as follows:

1. Both players put a $100 ante into the pot and receive one card. For simplicity, suppose that each player’s “card” is actually a random number uniformly distributed on [0,1].
2. Both players then simultaneously decide whether to “raise” or “stay”. Those who raise must put an additional $100 into the pot.
3. If anyone raises, the winner is the player with the highest card among those who raised. Or, if everyone stays, the winner is the player with the highest card.

Challenge: Find a strategy that *guarantees* you will earn a non-negative expected payoff, no matter what the other player does.

Game theory teaches us that, in any symmetric zero-sum game, the *Nash equilibrium strategy* guarantees a non-negative expected payoff. (Given symmetry, each player must earn zero expected payoff in any Nash equilibrium. Put differently, zero is the best expected payoff that your opponent can hope to achieve – by any strategy whatsoever – if you play the Nash equilibrium.) We may therefore reframe this challenge as finding a (symmetric) Nash equilibrium.

Challenge, Rephrased: Find a symmetric Nash equilibrium of World’s Simplest Poker.

<u>FALSE Conjecture</u>: An equilibrium exists with threshold T* such that (i) each player raises given any card above T* and (ii) each player stays given any card below T*.

<u>Proof that this is false</u>: Suppose that players adopt the conjectured strategies. Consider a player whose card is equal to the threshold, who I will call "the T*-type". First, suppose that the other player's card is below T*. Since the other player has a lower card and always stays, the T*-type always wins $100 whether he raises or stays. Next, suppose that the other player's card is above T*. Since the other player has a higher card and always raises, the T*-type always loses whether he raises or stays. However, he loses $100 more when raising in this case. Overall, then, the T*-type strictly prefers to stay, a contradiction.[1] QED

<u>TRUE Conjecture</u>: An equilibrium exists with a threshold T* and probability P* such that (i) each player raises given any card higher than T* and (ii) each player raises with probability P* given any card lower than T*.

<u>Proof that this is true</u>: Suppose that players adopt the conjectured strategies. Consider the incentives of the threshold type with card T*. First, suppose that the other player's card is above T*; this occurs with probability (1-T*). The threshold type loses no matter whether he raises or stays, but loses an extra $100 when raising. Overall, the "cost of raising" is 100 (1-T*) = 100 – 100T* in this case. Next, suppose that the other player's card is below T*; this occurs with probability T* and, in this case, the other player raises with probability P*. If the other player stays, the threshold type wins $100 whether he raises or stays. However, if the other player raises, the threshold type wins $200 instead

[1] Raising is a best response given slightly higher cards, while staying is a best response given slightly lower cards. This is only possible if, at the threshold, players are indifferent between raising or staying.

of losing \$100 when he raises instead of staying. Overall, the "benefit of raising" is 300 T* P* in this case.

At the threshold, the cost and benefit of raising must be the same, i.e. 100 = 100 T* (1 + 3P*) or, equivalently, 1 + 3P* = 1/T*. In fact, according to the conjecture, such indifference must hold everywhere below T*, not just at the threshold itself. Consider a player with card T* - x. Differences for this type when raising or staying, relative to the T*-threshold type, are as follows:

- When raising, T*-x type loses \$200 rather than winning \$200 when (i) the other player's card lies in [T*-x,T*] and (ii) the other player raises. Since this happens with probability P*x, the T*-x type's payoff when raising is 400P*x lower.
- When staying, T*-x type loses \$100 rather than winning \$100 when (i) the other player's card lies in [T*-x,T*] and (ii) the other player stays. The T*-x type's payoff when staying is therefore 200(1-P*)x lower.

Note that, as long as 2P* = (1-P*), or **P* = 1/3**, the T*-x type is indifferent between raising and staying for *all* x. Thus, randomizing is indeed a best response given any card less T*. Given P* = 1/3, the relationship 1+3P* = 1/T* implies **T* = 1/2**.

Lastly, we need to check that players prefer to raise given any card greater than T*. Consider a player with card T* + x. Differences for this type when raising or staying, relative to the T*-threshold type, are as follows:

- When raising, T*+x type wins \$200 rather than losing \$200 when the other player's card lies in [T*,T*+x] (given which the other player always raises). The T*+x type's payoff when raising is therefore 400x higher.

- When staying, T*+x type wins in exactly the same event as the T*-threshold type, namely, when (i) the other player's card is less than T* and (ii) the other player stays. Thus, the T*+x type's payoff when staying is the same.

Since the payoff from raising is increasing over [T*,1] and the payoff from staying is constant, each player clearly *strictly* prefers to raise given any card above T*. This completes the verification that the conjectured strategies constitute an equilibrium. QED